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Monge - form surfaces, $z = f(x, y)$

Part 1 - The basic general relationships

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1. Introduction

It is natural for meteorologists to think of a smooth hilly area as a Monge - form surface as this is a form which is convenient for them. Not so for the wizards of differential geometry. The Weatherburns, Grausteins, Struiks and Eisenharts of the mathematical world have naturally tended to prefer a greater generality and the Monge - form may crop up only in one or two irritating "exercises". Thus Eisenhart's well-known work "An introduction to Differential Geometry" mentions the Monge - form in only two places in the text and in two exercises. Forsythe's massive work provides just one exercise as coverage of this convenient form.

None of this will stop a determined meteorologist from finding out or working out what he wants to know about the Monge-form, but it does mean that he will be involved in much combing through the literature and much study of a lot of material not directly related to his immediate need. The purpose of this Note is to alleviate this tedious circumstance by providing the necessary formalism for dealing directly with $z = f(x, y)$. It is not intended to be exhaustive but it probably provides all that will be generally needed and will enable the rest to be easily worked out.

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2. Two-parameter surfaces in general - elementary fundamentals

A topographical surface can be defined by a position vector $\underline{r}$ which is a function of two independent parameters ϕ, ψ. The vector $\frac{\partial \underline{r}}{\partial \phi}$ is then tangential to the curve ψ = const. on the surface, whilst the vector $\frac{\partial \underline{r}}{\partial \psi}$ is tangential to the curve ϕ = const. on the surface. The vectors $\frac{\partial \underline{r}}{\partial \phi}$ and $\frac{\partial \underline{r}}{\partial \psi}$ in fact define the tangential plane to the surface at the point under consideration.

If we consider two neighbouring points $\underline{r}$ and $\underline{r} + \delta\underline{r}$ on the surface corresponding to parameter values ϕ, ψ and $\phi + d\phi, \psi + d\psi$ then

$$d\underline{r} = \frac{\partial \underline{r}}{\partial \phi} d\phi + \frac{\partial \underline{r}}{\partial \psi} d\psi \tag{1}$$

and the square of the length of the elemental arc of the curve joining these two points is, to the first order

$$ds^2 = d\underline{r} \cdot d\underline{r} = \left(\frac{\partial \underline{r}}{\partial \phi} d\phi + \frac{\partial \underline{r}}{\partial \psi} d\psi\right) \cdot \left(\frac{\partial \underline{r}}{\partial \phi} d\phi + \frac{\partial \underline{r}}{\partial \psi} d\psi\right) \tag{2}$$

i.e.

$$ds^2 = \left(\frac{\partial \underline{r}}{\partial \phi}\right)^2 d\phi^2 + 2\frac{\partial \underline{r}}{\partial \phi} \cdot \frac{\partial \underline{r}}{\partial \psi} d\phi d\psi + \left(\frac{\partial \underline{r}}{\partial \psi}\right)^2 d\psi^2 \tag{3}$$

It is then customary to write

$$E = \left(\frac{\partial \underline{r}}{\partial \phi}\right)^2 , \quad F = \frac{\partial \underline{r}}{\partial \phi} \cdot \frac{\partial \underline{r}}{\partial \psi} , \quad G = \left(\frac{\partial \underline{r}}{\partial \psi}\right)^2 \tag{4}$$

and to express (3) as

$$ds^2 = E d\phi^2 + 2F d\phi d\psi + G d\psi^2 \tag{5}$$

the quantities E, F and G being known as <u>the fundamental magnitudes of the first order</u>. The related quantity

$$H^2 = EG - F^2 \tag{6}$$

is also useful. It should be noted at once that for real ϕ and ψ the quantity $EG - F^2$ is always positive.

Considering now the second derivatives of $\underline{r}$, namely $\frac{\partial^2 \underline{r}}{\partial \phi^2}$, $\frac{\partial^2 \underline{r}}{\partial \phi \partial \psi}$ and $\frac{\partial^2 \underline{r}}{\partial \psi^2}$, the quantities

$$L = \underline{B} \cdot \frac{\partial^2 \underline{r}}{\partial \phi^2} , \quad M = \underline{B} \cdot \frac{\partial^2 \underline{r}}{\partial \phi \partial \psi} , \quad N = \underline{B} \cdot \frac{\partial^2 \underline{r}}{\partial \psi^2} \tag{7}$$

where $\underline{B}$ is the unit normal to the surface, are known as <u>the fundamental magnitudes of the second order</u>. It is also convenient to denote

$$T^2 = LN - M^2 \tag{8}$$

though, unlike H^2, this quantity is not necessarily positive.

These second order magnitudes are connected with the curvature of the surface. If we are considering a point P, position vector $\underline{r}$, then the position vector of a neighbouring point Q, to the second order terms is

$$\underline{r} + \left(\frac{\partial \underline{r}}{\partial \phi} d\phi + \frac{\partial \underline{r}}{\partial \psi} d\psi\right) + \frac{1}{2}\left[\frac{\partial^2 \underline{r}}{\partial \phi^2} d\phi^2 + 2\frac{\partial^2 \underline{r}}{\partial \phi \partial \psi} d\phi d\psi + \frac{\partial^2 \underline{r}}{\partial \psi^2} d\psi^2\right] \qquad (9)$$

and therefore the vector PQ is

$$\delta \underline{r} = \frac{\partial \underline{r}}{\partial \phi} d\phi + \frac{\partial \underline{r}}{\partial \psi} d\psi + \frac{1}{2}\left[\frac{\partial^2 \underline{r}}{\partial \phi^2} d\phi^2 + 2\frac{\partial^2 \underline{r}}{\partial \phi \partial \psi} d\phi d\psi + \frac{\partial^2 \underline{r}}{\partial \psi^2} d\psi^2\right] \qquad (10)$$

and the projection PR of PQ onto the normal $\underline{B}$ at P is thus

$$PR = \underline{B} \cdot \left(\frac{\partial \underline{r}}{\partial \phi} d\phi + \frac{\partial \underline{r}}{\partial \psi} d\psi\right) + \frac{1}{2} \underline{B} \cdot \left[\frac{\partial^2 \underline{r}}{\partial \phi^2} d\phi^2 + 2\frac{\partial^2 \underline{r}}{\partial \phi \partial \psi} d\phi d\psi + \frac{\partial^2 \underline{r}}{\partial \psi^2} d\psi^2\right] \qquad (11)$$

But $\frac{\partial \underline{r}}{\partial \phi}$ and $\frac{\partial \underline{r}}{\partial \psi}$ are in the tangent plane at P and are therefore at right angles to $\underline{B}$. Thus the first term in (11) vanishes, and the projection of PQ on to the normal at P is

$$PR = \underline{B} \cdot \delta \underline{r} = \frac{1}{2} \underline{B} \cdot \left[\frac{\partial^2 \underline{r}}{\partial \phi^2} d\phi^2 + 2\frac{\partial^2 \underline{r}}{\partial \phi \partial \psi} d\phi d\psi + \frac{\partial^2 \underline{r}}{\partial \psi^2} d\psi^2\right] \qquad (12)$$

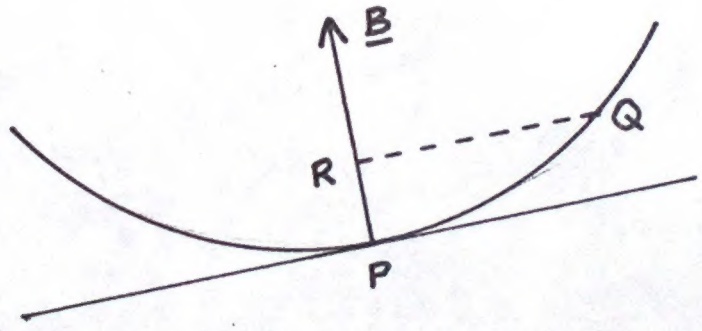


Figure 1

The situation is shown in Figure 1 which represents a section of the surface along the curve. Using (7) we have

$$PR = \underline{B} \cdot \delta \underline{r} = \frac{1}{2}\left[L d\phi^2 + 2M d\phi d\psi + N d\psi^2\right] \qquad (13)$$

and thus L, M, and N give the extent to which the surface bends away from the tangent plane at P over the interval PQ.

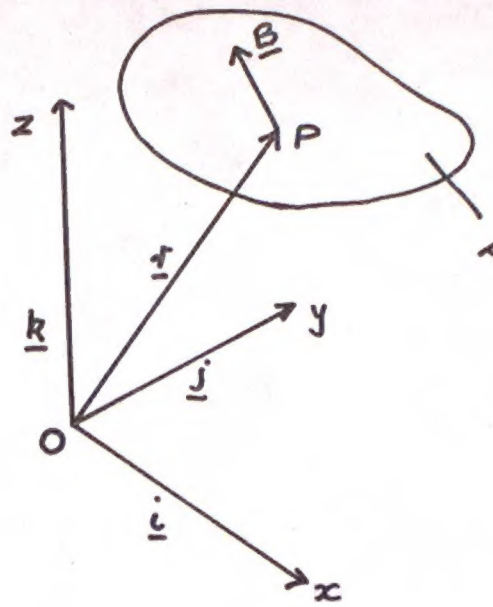
3. Monge form surfaces in particular — $z = f(x, y)$

In Section 2 the parameters ϕ, ψ have been taken to be quite general and can be any meaningful quantities. We now take them to be x, y the usual plane cartesian coordinates. As is customary in many texts we will define the quantities p. q, r, s, t to be

$$p = \frac{\partial f}{\partial x}, \quad q = \frac{\partial f}{\partial y}, \quad r = \frac{\partial^2 f}{\partial x^2}, \quad s = \frac{\partial^2 f}{\partial x \partial y}, \quad t = \frac{\partial^2 f}{\partial y^2} \qquad (14)$$

It is important at the outset to have a correct visualization of the situation and it would be helpful at this point if I could achieve one of those beautiful half-tone

drawings which adorn so many American text-books there. Alas, the best I can do for you is Figure 2.



surface $z = f(x, y)$ (15)

Figure 2

x, y, and z are the customary cartesian axes and $\underline{i}$, $\underline{j}$ and $\underline{k}$ are the customary associated unit vectors. The parametric curves on the surface $z = f(x, y)$ are the intersections of the planes $x = \text{const.}$ and $y = \text{const.}$ with the surface. The position vector $\underline{r}$ of any point P on the surface $z = f(x, y)$ is, as always,

$$\underline{r} = x\underline{i} + y\underline{j} + z\underline{k} \tag{16}$$

and by virtue of (15) this is also expressible as

$$\underline{r} = x\underline{i} + y\underline{j} + f(x, y)\,\underline{k} \tag{17}$$

Thus from (17)

$$\frac{\partial \underline{r}}{\partial x} = \underline{i} + \frac{\partial f(x, y)}{\partial x}\underline{k} \tag{18}$$

which we will rewrite as

$$\frac{\partial \underline{r}}{\partial x} = \underline{i} + \frac{\partial f}{\partial x}\underline{k} \tag{19}$$

or, using the first of (14)

$$\frac{\partial \underline{r}}{\partial x} = \underline{i} + p\underline{k} \tag{20}$$

It follows immediately that for the first order fundamental magnitude E we have

$$E = \left(\frac{\partial \underline{r}}{\partial x}\right)^2 = \frac{\partial \underline{r}}{\partial x} \cdot \frac{\partial \underline{r}}{\partial x} = (\underline{i} + p\underline{k}) \cdot (\underline{i} + p\underline{k})$$

i.e.

$$E = 1 + p^2 \tag{21}$$

and in a similar manner we can obtain

$$F = pq \tag{22}$$

and

$$G = 1 + q^2 \tag{23}$$

yielding also, from (6)

$$H^2 = 1 + p^2 + q^2 \tag{24}$$

Since $\frac{\partial \underline{r}}{\partial x}$ and $\frac{\partial \underline{r}}{\partial y}$ are vectors in the tangent plane to $z = f(x, y)$ at P it follows that the vector product $\frac{\partial \underline{r}}{\partial x} \times \frac{\partial \underline{r}}{\partial y}$ is a vector parallel to the normal to the surface at P. Therefore the unit normal $\underline{B}$ is obtained by virtue of the fact that

$$\frac{\partial \underline{r}}{\partial x} \times \frac{\partial \underline{r}}{\partial y} = \left| \frac{\partial \underline{r}}{\partial x} \times \frac{\partial \underline{r}}{\partial y} \right| \underline{B} \tag{25}$$

whence

$$\underline{B} = \frac{\frac{\partial \underline{r}}{\partial x} \times \frac{\partial \underline{r}}{\partial y}}{\left| \frac{\partial \underline{r}}{\partial x} \times \frac{\partial \underline{r}}{\partial y} \right|} \tag{26}$$

Now $\frac{\partial \underline{r}}{\partial x}$ is given by (20), and from (17)

$$\frac{\partial \underline{r}}{\partial y} = \underline{j} + q\underline{k} \tag{27}$$

so

$$\frac{\partial \underline{r}}{\partial x} \times \frac{\partial \underline{r}}{\partial y} = (\underline{i} + p\underline{k}) \times (\underline{j} + q\underline{k}) = -p\underline{i} - q\underline{j} + \underline{k} \tag{28}$$

and

$$\left| \frac{\partial \underline{r}}{\partial x} \times \frac{\partial \underline{r}}{\partial y} \right| = \sqrt{1 + p^2 + q^2} \tag{29}$$

and so, for the normal $\underline{B}$, from (29), (28), and (26) we have

$$\underline{B} = \frac{-p\underline{i} - q\underline{j} + \underline{k}}{\sqrt{1 + p^2 + q^2}} \tag{30}$$

From (21), (22), (23), it follows by referring back to (5) that the square of an element of arc on the surface $z = f(x, y)$ is

$$ds^2 = (1 + p^2)dx^2 + 2pq\,dx\,dy + (1 + q^2)dy^2 \tag{31}$$

From (18), differentiating again with respect to x there follows

$$\frac{\partial^2 \underline{r}}{\partial x^2} = \frac{\partial^2 f}{\partial x^2}\underline{k} = r\underline{k} \tag{32}$$

and therefore from the first of (7)

$$L = \underline{B} \cdot \frac{\partial^2 \underline{r}}{\partial x^2} = \frac{-p\underline{i} - q\underline{j} + \underline{k}}{\sqrt{1 + p^2 + q^2}} \cdot + \underline{k}$$

ie

$$L = \frac{r}{H} \tag{33}$$

and similarly from (7) and (30) the remaining fundamental magnitudes of the second order are found to be

$$M = \frac{s}{H} \tag{34}$$

and

$$N = \frac{t}{H} \tag{35}$$

and consequently, from (8)

$$T^2 = \frac{rt - s^2}{H^2} \tag{36}$$

Harking back to the remarks between (8) and (13) in Section 2, it is now seen that if Figure 1 is a section of a Monge form surface then

$$PR = \frac{1}{2H} (r\,dx + 2s\,dxdy + t\,dy) \tag{37}$$

4. <u>The parametric unit vectors a & b in the tangent plane</u>

If we look at Figure 2 in the positive direction of the y - axis and consider the intersection of the surface $z = f(x, y)$ by a plane through P parallel to the zx plan the situation is as in Figure 3

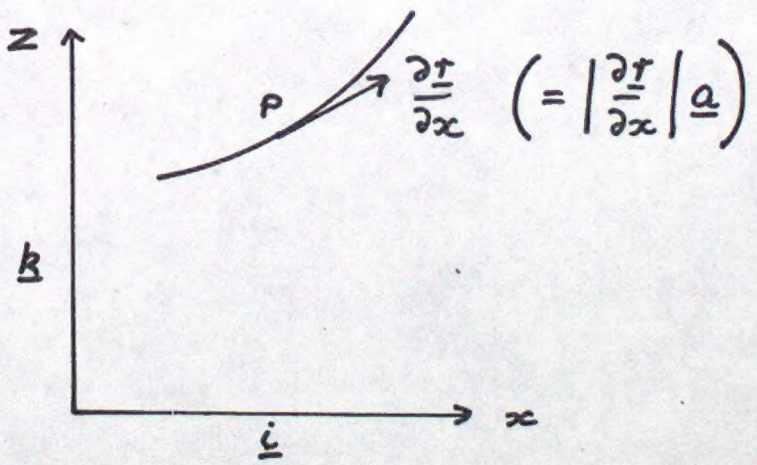


Figure 3

Similarly, looking at Figure 2 in the positive direction of the x - axis, the intersection of the surface by a plane through P parallel to the zy plane the situation is as in Figure 4.

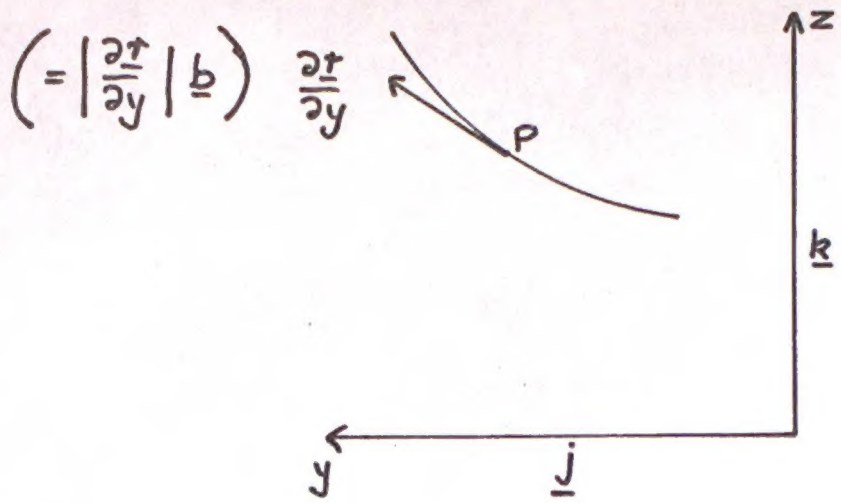


Figure 4

Now if the surface in Figure 2 is viewed looking down the normal $\underline{B}$ towards P the situation is as depicted in Figure 5.

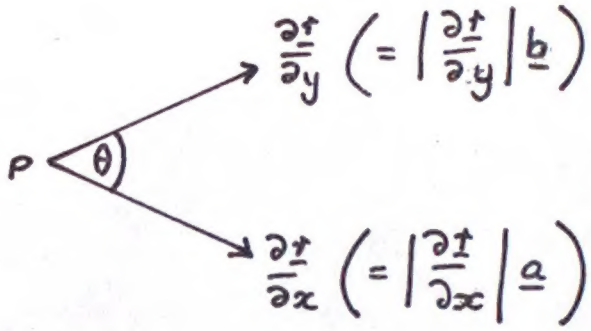


Figure 5

From Figures 2, 3, 4 and 5 and the foregoing equations some simple facts are made evident –

(i) $\frac{\partial \underline{r}}{\partial x}$ is not parallel to $\underline{i}$. $\frac{\partial \underline{r}}{\partial y}$ is not parallel to $\underline{j}$. Equations (20) and (27) show this. $\frac{\partial \underline{r}}{\partial x}$ and $\frac{\partial \underline{r}}{\partial y}$ are vectors in the tangent plane to the surface at P and they are tangent to the parametric curves on the surface $z = f(x, y)$, in this case the intersections of planes parallel to the zx and zy planes.

(ii) $\frac{\partial \underline{r}}{\partial x}$ and $\frac{\partial \underline{r}}{\partial y}$ are not unit vectors. If the unit vectors in these directions are respectively $\underline{a}$ and $\underline{b}$, then

$$\frac{\partial \underline{r}}{\partial x} = \left|\frac{\partial \underline{r}}{\partial x}\right|\underline{a} \quad , \quad \frac{\partial \underline{r}}{\partial y} = \left|\frac{\partial \underline{r}}{\partial y}\right|\underline{b} \tag{38}$$

From (20)

$$\left|\frac{\partial \underline{r}}{\partial x}\right| = 1 + p^2 \tag{39}$$

and from (27)

$$\left|\frac{\partial \underline{r}}{\partial y}\right| = 1 + q^2 \tag{40}$$

The unit vectors $\underline{a}$ and $\underline{b}$ are thus

$$\underline{a} = (1+p^2)^{-\frac{1}{2}} \frac{\partial \underline{r}}{\partial x} \quad , \quad \underline{b} = (1+q^2)^{-\frac{1}{2}} \frac{\partial \underline{r}}{\partial y} \tag{41}$$

or

$$\underline{a} = \frac{\underline{i} + p\underline{k}}{\sqrt{1+p^2}} \quad , \quad \underline{b} = \frac{\underline{j} + q\underline{k}}{\sqrt{1+q^2}} \tag{42}$$

and of course they are in the tangent plane.

(iii) $\frac{\partial \underline{r}}{\partial x}$ and $\frac{\partial \underline{r}}{\partial y}$ are <u>not</u> at right angles. $\underline{a}$ and $\underline{b}$ are <u>not</u> at right angles. The angle between them in the tangent plane, see Figure 6, is given by the scalar product of $\underline{a}$ and $\underline{b}$

$$\cos\theta = \underline{a} \cdot \underline{b} = \frac{pq}{\sqrt{(1+p^2)(1+q^2)}} \tag{43}$$

by using (41), (20), and (27). Then because

$$\underline{a} \times \underline{b} = \sin\theta \, \underline{B} \tag{44}$$

we have, from (42) and (30)

$$\sin\theta = \underline{a} \times \underline{b} \cdot \underline{B} = \sqrt{\frac{1+p^2+q^2}{(1+p^2)(1+q^2)}} \tag{45}$$

and thus

$$\tan\theta = \frac{\sqrt{1+p^2+q^2}}{pq} \tag{46}$$

5. The element of area dS on the surface $z = f(x, y)$

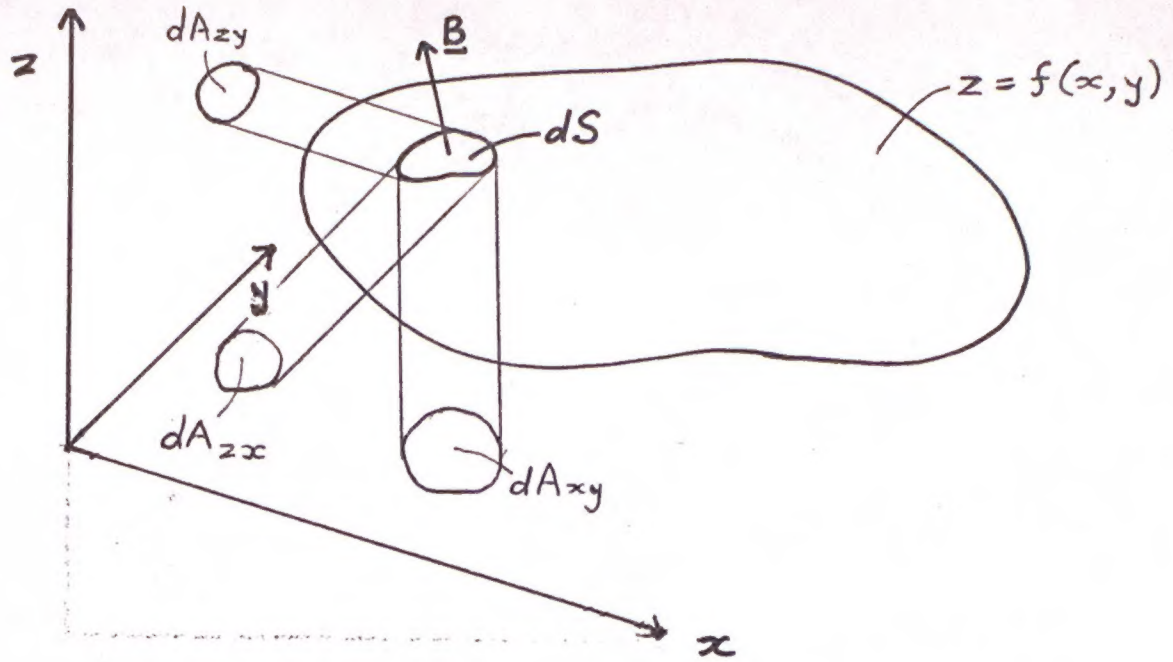


Figure 6

If the angles made by the normal $\underline{B}$ with the x, y, and z axes are $\alpha, \beta, \& \gamma$ respectively then

$$\underline{B} = \cos\alpha\,\underline{i} + \cos\beta\,\underline{j} + \cos\gamma\,\underline{k} \tag{47}$$

and by comparison with (30) it is straightaway evident that

$$\cos\alpha = \frac{-p}{\sqrt{1 + p^2 + q^2}} \tag{48}$$

$$\cos\beta = \frac{-q}{\sqrt{1 + p^2 + q^2}} \tag{49}$$

$$\cos\gamma = \frac{1}{\sqrt{1 + p^2 + q^2}} \tag{50}$$

and, of course

$$dA_{yz} = \cos\alpha\, dS \;,\; dA_{zx} = \cos\beta\, dS \;,\; dA_{xy} = \cos\gamma\, dS \tag{51}$$

A principle use of (48), (49), (50) and (51) is that if ϕ (x, y, z) is some function

defined on z = f (x, y) then its surface integral is

$$\mathcal{I} = \int_S \phi(x,y,z)\,dS \tag{52}$$

and it may be convenient to express this integral in any of the forms

$$\mathcal{I} = \int_{A_{yz}} \frac{\sqrt{1+p^2+q^2}}{-p}\,\phi(x,y,z)\,dy\,dz \tag{53}$$

or

$$\mathcal{I} = \int_{A_{zx}} \frac{\sqrt{1+p^2+q^2}}{-q}\,\phi(x,y,z)\,dz\,dx \tag{54}$$

or

$$\mathcal{I} = \int_{A_{xy}} \sqrt{1+p^2+q^2}\,\phi(x,y,z)\,dx\,dy \tag{55}$$

The element of area dS itself is given by the inverse of any of (51), most familiarly as

$$dS = \sec\gamma\, dA_{xy}$$

ie

$$dS = \sqrt{1+p^2+q^2}\,dx\,dy \tag{56}$$

6. The derivatives of a and b

First, from (20) and (27), we obtain

$$\frac{\partial^2 \underline{r}}{\partial x^2} = +\underline{k} \tag{30}$$

$$\frac{\partial^2 \underline{r}}{\partial x \partial y} = s\underline{k} \tag{57}$$

$$\frac{\partial^2 \underline{r}}{\partial y^2} = t\underline{k} \tag{58}$$

and since any vector can be expressed in terms of any three non-coplanar vectors we can express the vectors on the left hand sides of (30), (57) and (58) in terms of

$\frac{\partial \underline{r}}{\partial x}$, $\frac{\partial \underline{r}}{\partial y}$, and $\underline{B}$, as

$$\frac{\partial^2 \underline{r}}{\partial x^2} = \alpha_1 \frac{\partial \underline{r}}{\partial x} + \beta_1 \frac{\partial \underline{r}}{\partial y} + \gamma_1 \underline{B} \tag{59}$$

$$\frac{\partial^2 \underline{r}}{\partial x \partial y} = \alpha_2 \frac{\partial \underline{r}}{\partial x} + \beta_2 \frac{\partial \underline{r}}{\partial y} + \gamma_2 \underline{B} \tag{60}$$

$$\frac{\partial^2 \underline{r}}{\partial y^2} = \alpha_3 \frac{\partial \underline{r}}{\partial x} + \beta_3 \frac{\partial \underline{r}}{\partial y} + \gamma_3 \underline{B} \tag{61}$$

where the α_i , β_i , γ_i are just coefficients (nothing to do with α, β, γ in (48), (49), and (50)). And now by taking the scalar products of $\underline{B}$, $\frac{\partial \underline{r}}{\partial x}$, + $\frac{\partial \underline{r}}{\partial y}$ through (59), (60), and (61) in turn and using the equations already mentioned in this section it is soon established that

$$\alpha_1 = \frac{pt}{H^2} \quad , \quad \alpha_2 = \frac{ps}{H^2} \quad , \quad \alpha_3 = \frac{pt}{H^2} \tag{62}$$

$$\beta_1 = \frac{qt}{H^2} \quad , \quad \beta_2 = \frac{qs}{H^2} \quad , \quad \beta_3 = \frac{qt}{H^2} \tag{63}$$

$$\gamma_1 = L = \frac{t}{H} \quad , \quad \gamma_2 = M = \frac{s}{H} \quad , \quad \gamma_3 = N = \frac{t}{H} \tag{64}$$

Furthermore from (41) it can be shown that

$$\frac{\partial^2 \underline{r}}{\partial x^2} = pt(1+p^2)^{-\frac{1}{2}} \underline{a} + (1+p^2)^{+\frac{1}{2}} \frac{\partial \underline{a}}{\partial x} \tag{65}$$

$$\frac{\partial^2 \underline{r}}{\partial y \partial x} = ps(1+p^2)^{-\frac{1}{2}} \underline{a} + (1+p^2)^{\frac{1}{2}} \frac{\partial \underline{a}}{\partial y} \tag{66}$$

$$\frac{\partial^2 \underline{r}}{\partial x \partial y} = qs(1+q^2)^{-\frac{1}{2}} \underline{b} + (1+q^2)^{\frac{1}{2}} \frac{\partial \underline{b}}{\partial x} \tag{67}$$

$$\frac{\partial^2 \underline{r}}{\partial y^2} = qt(1+q^2)^{-\frac{1}{2}} \underline{b} + (1+q^2)^{\frac{1}{2}} \frac{\partial \underline{b}}{\partial y} \tag{68}$$

so that we are finally able to convert (59), (60), (61) into

$$\frac{\partial \underline{a}}{\partial x} = -\frac{rpq^2}{H^2(1+p^2)}\,\underline{a} + \frac{rq}{H^2}\left(\frac{1+q^2}{1+p^2}\right)^{\frac{1}{2}}\underline{b} + \frac{r}{H\sqrt{1+p^2}}\,\underline{B} \qquad (69)$$

$$\frac{\partial \underline{a}}{\partial y} = -\frac{spq^2}{H^2(1+p^2)}\,\underline{a} + \frac{sq}{H^2}\left(\frac{1+q^2}{1+p^2}\right)^{\frac{1}{2}}\underline{b} + \frac{s}{H\sqrt{1+p^2}}\,\underline{B} \qquad (70)$$

$$\frac{\partial \underline{b}}{\partial x} = \frac{sp}{H^2}\left(\frac{1+p^2}{1+q^2}\right)^{\frac{1}{2}}\underline{a} - \frac{sqp^2}{H^2(1+q^2)}\,\underline{b} + \frac{s}{H\sqrt{1+q^2}}\,\underline{B} \qquad (71)$$

$$\frac{\partial \underline{b}}{\partial y} = \frac{tp}{H^2}\left(\frac{1+p^2}{1+q^2}\right)^{\frac{1}{2}}\underline{a} - \frac{tqp^2}{H^2(1+q^2)}\,\underline{b} + \frac{t}{H\sqrt{1+q^2}}\,\underline{B} \qquad (72)$$

The reader may wonder how it is that $\frac{\partial \underline{a}}{\partial x}$, for example, has a contribution in the $\underline{a}$ direction. The reason is that we are dealing with an oblique parametric net on the surface. $\frac{\partial \underline{a}}{\partial x}$ is indeed at right angles to $\underline{a}$, but it is not in the direction of $\underline{b}$. This is a point which may not be readily accepted by anybody who is not familiar with the quirks of oblique coordinate systems, so it may help to demonstrate that (69), for instance, gives sensible results. Take the scalar product of $\underline{a}$ through (69). On the LHS we have $\underline{a} \cdot \frac{\partial \underline{a}}{\partial x} = 0$. On the RHS $\underline{a}.\underline{a} = 1$ and $\underline{a}.\underline{B} = 0$ (since $\underline{a}$ is in the tangent plane and $\underline{B}$ is normal to it).

Thus we are left with

$$0 = -\frac{rpq^2}{H^2(1+p^2)} + \frac{rq}{H^2}\left(\frac{1+q^2}{1+p^2}\right)\underline{a}\cdot\underline{b}$$

and on doing the necessary cancelling and rearranging it is found that this comes to precisely (43).

By using (42) the equations (69) to (72) may be expressed as

$$\frac{\partial \underline{a}}{\partial x} = \frac{r}{(1+p^2)^{3/2}}(-p\underline{i} + \underline{k}) \qquad (73)$$

$$\frac{\partial \underline{a}}{\partial y} = \frac{s}{(1+p^2)^{3/2}}(-p\underline{i} + \underline{k}) \qquad (74)$$

$$\frac{\partial \underline{b}}{\partial x} = \frac{s}{(1+q^2)^{3/2}}(-q\underline{i} + \underline{k}) \qquad (75)$$

$$\frac{\partial \underline{b}}{\partial y} = \frac{t}{(1+q^2)^{3/2}}(-q\underline{j} + \underline{k}) \qquad (76)$$

Also needed for some purposes is the change of the angle θ between the parametric lines on the surface along the parametric directions. These may be obtained either by differentiating (43) or by the use of (69) to (72). For example, the rate of change of θ in the x - direction is obtained from

$$\frac{\partial}{\partial x}(\underline{a}\cdot\underline{b}) = \underline{a}\cdot\frac{\partial \underline{b}}{\partial x} + \underline{b}\cdot\frac{\partial \underline{a}}{\partial x} \qquad (77)$$

followed by the use of (69) and (71). The results come out as

$$\frac{\partial \theta}{\partial x} = -\frac{1}{\sqrt{1+p^2+q^2}}\left(\frac{qr}{1+p^2} + \frac{ps}{1+q^2}\right) \qquad (78)$$

and

$$\frac{\partial \theta}{\partial y} = -\frac{1}{\sqrt{1+p^2+q^2}}\left(\frac{qs}{1+p^2} + \frac{pt}{1+q^2}\right) \qquad (79)$$

7. The derivatives of B

$\frac{\partial \underline{B}}{\partial x}$ and $\frac{\partial \underline{B}}{\partial y}$ are most simply found by the technique used by Weatherburn Vol 1,

pp 60 and 61, which leads to

$$H^2 \frac{\partial \underline{B}}{\partial x} = (FM - GL)\frac{\partial \underline{r}}{\partial x} + (FL - EM)\frac{\partial \underline{r}}{\partial y} \tag{80}$$

$$H^2 \frac{\partial \underline{B}}{\partial y} = (FN - GM)\frac{\partial \underline{r}}{\partial x} + (FM - EN)\frac{\partial \underline{r}}{\partial y} \tag{81}$$

whereupon, using (21), (22), (23), (24), (33), (34) and (35), we have

$$(1+p^2+q^2)^{\frac{3}{2}} \frac{\partial \underline{B}}{\partial x} = \left[pqs - (1+q^2)r\right]\frac{\partial \underline{r}}{\partial x} + \left[pqr - (1+p^2)s\right]\frac{\partial \underline{r}}{\partial y} \tag{82}$$

$$(1+p^2+q^2)^{\frac{3}{2}} \frac{\partial \underline{B}}{\partial y} = \left[pqt - (1+q^2)s\right]\frac{\partial \underline{r}}{\partial x} + \left[pqs - (1+p^2)t\right]\frac{\partial \underline{r}}{\partial y} \tag{83}$$

and these may be expressed in terms of $\underline{a}$ and $\underline{b}$ by using (41) to get

$$\frac{\partial \underline{B}}{\partial x} = \frac{\sqrt{1+p^2}\left[pqs - (1+q^2)r\right]\underline{a} + \sqrt{1+q^2}\left[pqr - (1+p^2)s\right]\underline{b}}{(1+p^2+q^2)^{3/2}} \tag{84}$$

$$\frac{\partial \underline{B}}{\partial y} = \frac{\sqrt{1+p^2}\left[pqt - (1+q^2)s\right]\underline{a} + \sqrt{1+q^2}\left[pqs - (1+p^2)t\right]\underline{b}}{(1+p^2+q^2)^{3/2}} \tag{85}$$

or in terms of $\underline{i}$, $\underline{j}$, and $\underline{k}$ by using (20) and (27) to get

$$\frac{\partial \underline{B}}{\partial x} = \frac{\left[pqs - (1+q^2)r\right]\underline{i} + \left[pqr - (1+p^2)s\right]\underline{j} - (pr+qs)\underline{k}}{(1+p^2+q^2)^{3/2}} \tag{86}$$

$$\frac{\partial \underline{B}}{\partial y} = \frac{\left[pqt - (1+q^2)s\right]\underline{i} + \left[pqs - (1+p^2)t\right]\underline{j} - (ps+qt)\underline{k}}{(1+p^2+q^2)^{3/2}} \tag{87}$$

8. Comment

These are the basic relationships needed to deal with Monge-form surfaces. Some more specialized matters will be dealt with in subsequent Notes.

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